

Analysis of Viscoelastic Fluid with Heat Transfer in Helical Screw Rheometer

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ABSTRACT

This study presents an analytical investigation of nonlinear heat transfer in viscoelastic materials processed through a single screw extruder. The model is based on a viscoelastic fluid framework, which captures both elastic and viscous effects relevant to polymer melt behavior. A perturbation technique is applied to solve the governing nonlinear differential equations for temperature fields in the barrel, screw, and fluid domains. This study investigates nonlinear temperature effects, offering new insights into the thermal behavior under screw-driven extrusion. A thorough investigation is conducted on the influence of key parameters, including the pressure gradient, Weissenberg number (Wi), and Brinkman number (Br), on temperature and heat flux profiles. The results show that increasing Br from 1 to 5 significantly raises the peak temperature and flow characteristics, and are interpreted in terms of the underlying physical mechanisms.

1. Introduction

Extrusion is one of the most prevalent manufacturing processes in industries handling materials such as rubber, plastics, polymers, and pharmaceuticals. In screw extrusion, raw materials are fed into a hopper and conveyed through the annular space between a rotating screw and a motionless barrel. As the material progresses, it undergoes mechanical shear and pressure and is subjected to various transformations, eventually being forced through a die of predefined geometry to achieve the desired cross-sectional shape. Extrusion has also gained prominence in the food processing industry in recent years, where raw food ingredients are subjected to high shear rates and elevated temperatures. This environment enables intensive mixing and triggers chemical reactions that constitute the cooking process. Single-screw and twin-screw extruders have been widely used in food processing, with recent studies providing a

comprehensive review of the relative advantages of each configuration [1]. Additionally, extensive experimental and computational research has been conducted to elucidate the underlying physical mechanisms governing the thermal and rheological behaviour of polymeric materials during extrusion processes.

In industrial manufacturing processes, precise control of fluid rheological properties is crucial to achieve the desired product quality and shape. Various viscometers are used in industries to measure the rheological behaviour of different fluids. However, researchers state that traditional viscometers have limitations in capturing all fundamental physical properties of fluids. To overcome this limitation, Kraynik et. al. [2] developed alternative devices, known as the helical screw rheometer (HSR), a modern rotational mixer viscometer resembling a single screw extruder. The HSR consists of a helical screw within a tight-fitting cylinder, where the rotating

screw generates a pressure differential proportional to the fluid's viscosity, allowing for accurate measurement of rheological properties.

Non-Newtonian fluids have received considerable attention in recent research due to their diverse technological applications across engineering, industrial, and biological domains. The co-rotational Maxwell fluid model, a viscoelastic fluid paradigm, captures both viscous and elastic behaviours, exhibiting shear thinning, elasticity, and memory effects. This model is characterized by nonlinear relationships between shear stress and deformation rate. Viscoelastic fluids like co-rotational Maxwell fluids are crucial in understanding complex fluid dynamics and can effectively describe fluids under moderate to higher shear, typically in single-screw extruders. Compared to other viscoelastic fluids, the co-rotational Maxwell model is adequate for describing bulk polymer melts with moderate elastic effects. The assumption of constant fluid properties is commonly used in the literature to simplify flow behaviour analysis. However, real-world flows often experience temperature variations due to external heating or high shear rates, which significantly impact fluid properties, particularly viscosity. Heat transfer plays a vital role in energy and entropy transport, governed by mechanisms such as convection, radiation, and conduction.

Nowadays, researchers are interested in studying non-Newtonian viscoelastic fluids in the extrusion process. The previous work on the classical extrusion theory also encompasses the work of Kousemaker et. al. [3], Gupta V. et. al. [4]. Consequently, heat transfer, thermodynamics, and fluid dynamics play pivotal roles in governing these processes. This phenomenon may be observed in several industries such as wire coating, hot rolling, power generation, glass fibre, polymer sector, metal spinning, glass sheet manufacture, and paper production. Zeb, M. et. al. [5] examined the couple stress fluid with heat transfer in helical screw rheometer using perturbation method. It is of great importance to physicists, mathematicians, and in the industrial area due to their many technological applications such as biological, and for instance, clay coatings, greases, toothpaste, paints, blood, drilling mud, and polymer melts, etc., its study yields a lot of scientific insight to optimize the processing and improve the quality of production, regarding circular geometry [6, 7].

Solnar S. et. al. [8] examined the evolution of rheometers and their importance in characterizing the

viscosities of fluids. To make rheometry more accessible, researchers have investigated cost-effective alternatives, including the capillary-slit rheometer, with experimental validation using devices like VIEWR. The research [9] focused on modelling and simulating the melt conveying zones of single-screw extruders. The authors investigated fully developed, steady-state laminar flow of incompressible Newtonian fluids with constant viscosity, deriving exact analytical solutions and conducting numerical studies to analyse the flow behaviour. The study [10] employed the two-relaxation-time lattice Boltzmann method to simulate the flow behaviour of generalized Newtonian fluids within a single-screw extruder. The research examined the effects of rheological properties and pressure gradients on extrusion efficiency by analysing velocity profiles and volumetric flow rates. Comparisons with analytical solutions demonstrated that the lattice Boltzmann method offers higher accuracy for modelling flows in such complex geometries. Xuesi Gao. et. al. [11] experimentally and numerically simulated the non-Newtonian shear thinning fluids to examine viscosity, rate, and characteristics of screw for non-Newtonian fluid in a rectangular channel. The work of Delver. E. et. al. [12] presents a previous review of single and double-screw extruders. The author focused on the geometry, modelling the mixing process in CFD software, and pressure and temperature modelling. They found and revealed that there is a discrepancy in the CFD model and analytic models, specifically at the outlet where the flow is restricted. Pricci, et. al. [13] had investigated flow non-Newtonian power law fluid in single screw extruders using semi analytical technique. They have calculated temperature, channel width, flow rate, and height of the extruder. They compared their results with numerical simulation and revealed that their computation fully agrees with the numerical results obtained.

When these fluids are studied experimentally, it costs a lot of time, resources, and money. Given this, researchers are keenly interested in theoretical analysis, which often results in nonlinear differential equations that are then solved using numerical methods, analytical methods, and integral transforms. Due to the complexity of numerical approaches, various analytical methods are used to overcome complexity and manage the nonlinearity of the model. These methods include the homotopy analysis method (HAM), the homotopy perturbation method (HPM), Adomian decomposition method (ADM), and the Variational Iterative method. Despite their utility,

these methods also have intricacies and limitations in handling strong nonlinearities.

This research work extends the study presented in [14], where the authors investigated the velocity profile, volumetric flow rate, shear stresses, and average velocity of a co-rotational Maxwell fluid. Building on this previous research, this study thoroughly investigates the heat transfer characteristics of a co-rotational Maxwell fluid within an extruder. Unlike previous work that primarily focuses on velocity, this study presents a comprehensive analysis of nonlinear heat transfer using the perturbation technique, which provides an innovative solution framework. The technique is a well-established method often employed to solve problems related to vibration, heat transport, fluid mechanics, and solid mechanics when the parameter is small. It can solve multidimensional, nonlinear, and inhomogeneous problems with a fairly high degree of accuracy [15].

2. Theoretical Framework of the Problem

The extrusion process involves conveying, melting, and extruding the polymer. Melting occurs in the transition zone, and to forecast the flow and heat transfer in the extruder accurately, it is crucial to model this process adequately. For ease of analysis and visualization, the coordinate system is set at the screw root; hence, the barrel rotates in the opposite direction of the screw. This arrangement is commonly used in the literature [16]. Curvature effects are significant in sections with deep channels, like the solid transporting zone. Nevertheless, in modelling the shallow metering section considered here, the impact of curvature is intentionally excluded. This makes it possible to open out the screw channel, creating the rectangular computing domain seen in Fig. 1, in which the barrel moves with velocity V at the screw helix angle θ . The barrel's total velocity has a component along the main flow direction. For simplicity and clarity, it is necessary to decompose the inclined motion into: U , representing the velocity along the x -axis, and W , representing the velocity along the z -axis. This decomposition provides a mathematically consistent and physically interpretable way to impose boundary conditions in a simplified Cartesian domain.

The screw flights have minimal influence on the flow due to the large width-to-depth ratio of the channel. This is analogous to the flow between parallel, infinite plates. This results in essentially two-dimensional flow, as transverse convection effects are negligible. For most polymer melts, when employing normal pressure, gravity effects are neglected, and density remains unaffected. Polymer melts exhibit high

viscosity, and extrusion operations involve slow and steady motion. Thus, inertia forces are relatively negligible compared to viscous forces and are therefore disregarded. These approximations are generally employed in polymer processing.

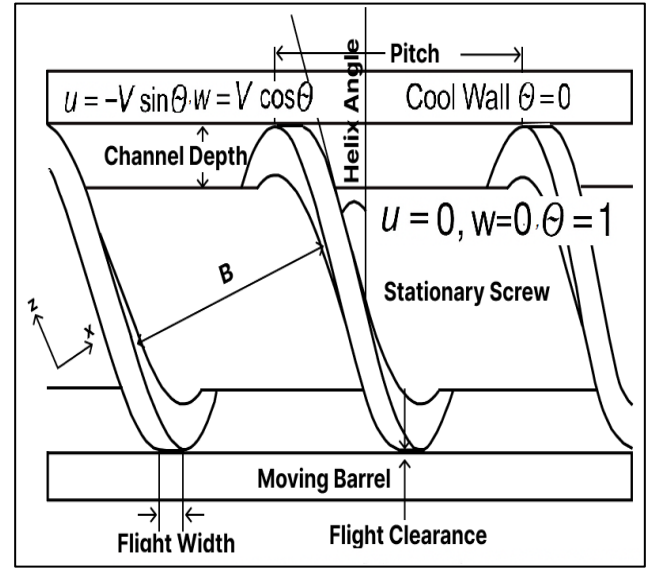


Fig. 1. The geometry of screw channel and barrel

For the co-rotational Maxwell fluid, the extra stress tensor is mentioned as follows:

$$\vec{\sigma} + \lambda_1 \sigma^\nabla + \frac{\lambda_1}{2} (\vec{A}_1 \vec{\sigma} + \vec{\sigma} \vec{A}_1) = \eta_0 \vec{A}_1. \quad (1)$$

Here λ_1 and η_0 are the relaxation time and zero shear viscosity, respectively. Furthermore, the contravariant derivative is $\sigma^\nabla = (\vec{\sigma} \cdot \nabla) \vec{\sigma} - \{(\nabla \vec{V})^T \vec{\sigma} + \vec{\sigma} (\nabla \vec{V})\}$, and the first Rivlin-Eriksen tensor is $\vec{A}_1 = (\nabla \vec{V}) + (\nabla \vec{V})^T$. Assuming a steady, two-dimensional, incompressible flow of a co-rotational Maxwell fluid in a single-screw extruder with a shallow channel in the x and z directions (i.e., see Fig. 1. $B \gg h$), then ignoring the edge effects, the momentum equation yields

$$\frac{\partial P}{\partial x} = \frac{\partial \sigma_{xy}}{\partial y}, \quad \frac{\partial P}{\partial y} = \frac{\partial \sigma_{yy}}{\partial y}, \quad \frac{\partial P}{\partial z} = \frac{\partial \sigma_{yz}}{\partial y}. \quad (2)$$

Here, P is the pressure. Given the preceding assumptions, the velocity field and extra stress tensor are described by

$$\vec{V} = [u(y), 0, w(y)], \quad \vec{\sigma} = \sigma(y). \quad (3)$$

Using assumption from equation (3) in equation (1), the succeeding statements are acquired as in equation 4.

$$\vec{A_1} = \begin{bmatrix} 0 & \frac{du}{dy} & 0 \\ \frac{du}{dy} & 0 & \frac{dw}{dy} \\ 0 & \frac{dw}{dy} & 0 \end{bmatrix}, \quad (4)$$

$$\hat{\sigma} = - \begin{bmatrix} 2\sigma_{xy} \frac{du}{dy} & \sigma_{yy} \frac{du}{dy} & \sigma_{yz} \frac{du}{dy} + \sigma_{xy} \frac{dw}{dy} \\ \sigma_{yy} \frac{du}{dy} & 0 & \sigma_{yy} \frac{dw}{dy} \\ \sigma_{yz} \frac{du}{dy} + \sigma_{xy} \frac{dw}{dy} & \sigma_{yy} \frac{dw}{dy} & 2\sigma_{zy} \frac{dw}{dy} \end{bmatrix}, \quad (5)$$

$$(\vec{A_1} \vec{\sigma} + \vec{\sigma} \vec{A_1}) = \begin{bmatrix} 2\sigma_{xy} \frac{du}{dy} & (\sigma_{yy} + \sigma_{xx}) \frac{du}{dy} + \sigma_{xz} \frac{dw}{dy} \\ (\sigma_{yy} + \sigma_{xx}) \frac{du}{dy} + \sigma_{xz} \frac{dw}{dy} & 2\sigma_{xy} \frac{du}{dy} + 2\sigma_{zy} \frac{dw}{dy} \\ \sigma_{yz} \frac{du}{dy} + \sigma_{xy} \frac{dw}{dy} & \sigma_{xz} \frac{du}{dy} + (\sigma_{yy} + \sigma_{zz}) \frac{dw}{dy} \\ \sigma_{xz} \frac{du}{dy} + (\sigma_{yy} + \sigma_{zz}) \frac{dw}{dy} & 2\sigma_{zy} \frac{dw}{dy} \\ 2\sigma_{zy} \frac{dw}{dy} \end{bmatrix}. \quad (6)$$

Where the normal and shear components are defined below

$$\sigma_{xx} = \lambda_1 \sigma_{xy} \frac{du}{dy}, \quad (7)$$

$$\sigma_{xy} = \frac{\eta_0 \frac{du}{dy}}{1 + \lambda_1^2 \left\{ \left(\frac{du}{dy} \right)^2 + \left(\frac{dw}{dy} \right)^2 \right\}}, \quad (8)$$

$$\sigma_{xz} = \frac{1}{2} \lambda_1 \left\{ \sigma_{yz} \frac{du}{dy} + \sigma_{xy} \frac{dw}{dy} \right\}, \quad (9)$$

$$\sigma_{yy} = -\lambda_1 \left\{ \sigma_{yz} \frac{dw}{dy} + \sigma_{xy} \frac{du}{dy} \right\}, \quad (10)$$

$$\sigma_{yz} = \frac{\eta_0 \frac{dw}{dy}}{1 + \lambda_1^2 \left\{ \left(\frac{du}{dy} \right)^2 + \left(\frac{dw}{dy} \right)^2 \right\}}, \quad (11)$$

$$\sigma_{zz} = -\lambda_1 \sigma_{yz} \frac{dw}{dy}. \quad (12)$$

Equation (2) conveys that P is a function of x and z , which demonstrates that P_x and P_z are non-variables. On explaining the reformed pressure as $\hat{P} = P - \sigma_{yy}$ and inserting equations (8), (9), and (11), we acquire

$$\frac{\partial \hat{P}}{\partial x} = \frac{d}{dy} \left[\frac{\eta_0 \left(\frac{du}{dy} \right)}{1 + \lambda_1^2 \left\{ \left(\frac{du}{dy} \right)^2 + \left(\frac{dw}{dy} \right)^2 \right\}} \right], \quad (13)$$

$$\frac{\partial \hat{P}}{\partial y} = 0, \quad (14)$$

$$\frac{\partial \hat{P}}{\partial z} = \frac{d}{dy} \left[\frac{\eta_0 \left(\frac{dw}{dy} \right)}{1 + \lambda_1^2 \left\{ \left(\frac{du}{dy} \right)^2 + \left(\frac{dw}{dy} \right)^2 \right\}} \right]. \quad (15)$$

The occurrence of significant viscous dissipation and/or internal heat generation leads to pronounced thermal convection in the z -direction, resulting in a non-uniform temperature distribution with a notable increase in the temperature field along the z -axis. Compared to convective transport, the spatial distribution in the z -direction is negligible. Nevertheless, the temperature variation in the y -direction is anticipated to be significantly more pronounced than the gradient observed in the downward flow direction within the channel. Consequently, the diffusive flux in the y -direction is preserved, ensuring a more accurate representation of thermal transport phenomena. Therefore,

$$\frac{1}{\rho C_p} \left[\kappa \frac{d^2 T}{dy^2} + \sigma_{yz} \frac{dw}{dy} + \sigma_{xy} \frac{du}{dy} \right] = 0. \quad (16)$$

The heat flux vector Q_y is further defined in terms of the temperature gradient as

$$Q_y = -\lambda \frac{\partial T}{\partial y}. \quad (17)$$

In the above equation, temperature is denoted by T , ρ is the density, specific heat is C_p , and κ represents the conductivity. The first term in equation (16) represents thermal diffusion, whereas the last two terms represent viscous dissipation. Substituting equations (8) and (11) into equation (16) yields

$$-\frac{\kappa}{\eta_0} \frac{d^2 T}{dy^2} \left[1 + \lambda_1^2 \left\{ \left(\frac{du}{dy} \right)^2 + \left(\frac{dw}{dy} \right)^2 \right\} \right] = \left\{ \left(\frac{du}{dy} \right)^2 + \left(\frac{dw}{dy} \right)^2 \right\}. \quad (18)$$

In our scenario, $T=T(y)$. The fluid within the channel is heated by the screw (at $y=0$) and the barrel (at $y=h$), which impose a constant temperature. This scenario gives rise to the following boundary conditions:

$$u = 0, w = 0, \quad T = T_i \quad \text{on} \quad y = 0, \quad (19)$$

$$u = U, \quad w = W, \quad T = T_b \quad \text{at} \quad y = h. \quad (20)$$

Here $U=-V\sin\theta$, and $W=V\cos\theta$. The aforementioned equations are rendered dimensionless by introducing the following characteristic quantities, yielding the familiar dimensionless parameters [6]:

$$x = x^*h, y = y^*h, u = u^*w, w = w^*W, P = P^*\mu\left(\frac{W}{h}\right), \varphi^* = \frac{(T-T_b)}{T_s-T_b}, Br = \frac{\eta_0 W^2}{\kappa(T_s-T_b)}, W_i^2 = \left(\frac{\lambda_1 W}{h}\right)^2, Q = \frac{hQ_y}{\kappa(T_s-T_b)}. \quad (21)$$

Here, $W_i = \lambda_1 W/h$ denotes the Weissenberg number. On introducing equation (21) in equation (18), the resultant equations after integration of equations (13) and (15), then by dropping ‘*’, we obtain

$$\frac{du}{dy} = (P_x y + A_1) \left[1 + (W_i)^2 \left\{ \left(\frac{du}{dy} \right)^2 + \left(\frac{dw}{dy} \right)^2 \right\} \right], \quad (22)$$

$$\frac{dw}{dy} = (P_z y + A_2) \left[1 + (W_i)^2 \left\{ \left(\frac{du}{dy} \right)^2 + \left(\frac{dw}{dy} \right)^2 \right\} \right], \quad (23)$$

$$\frac{d^2\varphi}{dy^2} = - \left[W_i^2 \frac{d^2\varphi}{dy^2} + Br \right] \left[\left(\frac{du}{dy} \right)^2 + \left(\frac{dw}{dy} \right)^2 \right]. \quad (24)$$

Here $P_x = \frac{\partial P}{\partial x}$, $P_z = \frac{\partial P}{\partial z}$; A_1 and A_2 are arbitrary constants of integration. The dimensionless heat flux is demarcated as:

$$Q = -\frac{\partial\varphi}{\partial y}. \quad (25)$$

And unit-less boundary conditions are

$$u = 0, w = 0, \varphi = 1 \quad \text{at } y = 0, \quad (26)$$

$$u = -\tan\theta = \frac{U}{W}, w = 1, \varphi = 0, \text{ at } y = 1. \quad (27)$$

3. Solution Methodology

The differential equations, derived above, are the nonlinear ordinary differential equations. Consequently, an analytical approximation approach, a perturbation method, which is reasonably straightforward and accurate, is applied in this piece of work. The perturbation theory relies on the existence of a small parameter in the equation, either explicitly or implicitly. This small parameter defines the process under examination. The perturbation approach is applicable for small parameters $\varepsilon \ll 1$, where the nonlinearity has a weak effect. However, it has been demonstrated to have good accuracy even for

relatively large values of the perturbation parameter ε [16].

Assuming perturbation parameter $\varepsilon = Wi^2$, and introducing it into equations (22) - (27), the dimensionless velocities and temperature flow variables are expanded up to the second-order approximation.

$$u(y; \varepsilon) = \varepsilon^0 u_0(y) + \varepsilon^1 u_1(y) + \varepsilon^2 u_2(y) + O(\varepsilon^3), \quad (28)$$

$$w(y; \varepsilon) = \varepsilon^0 w_0(y) + \varepsilon^1 w_1(y) + \varepsilon^2 w_2(y) + O(\varepsilon^3), \quad (29)$$

$$\varphi(y; \varepsilon) = \varepsilon^0 \varphi_0(y) + \varepsilon^1 \varphi_1(y) + \varepsilon^2 \varphi_2(y) + O(\varepsilon^3). \quad (30)$$

Comparing like powers of ε , we achieve the system of differential equations.

3.1 Zeroth Order Problem $O(\varepsilon^0)$ and its Solution

$$\frac{du_0}{dy} = (P_x y + A_1), \quad (31)$$

$$\frac{dw_0}{dy} = (P_z y + A_2), \quad (32)$$

$$\frac{d^2\varphi_0}{dy^2} = -Br \left[\left(\frac{du_0}{dy} \right)^2 + \left(\frac{dw_0}{dy} \right)^2 \right]. \quad (33)$$

Along the boundary conditions:

$$u_0 = 0, w_0 = 0, \text{ and } \varphi_0 = 1, \text{ at } y = 0, \quad (34)$$

$$u_0 = \frac{U}{W}, w_0 = 1, \text{ and } \varphi_0 = 0, \text{ at } y = 1. \quad (35)$$

Producing the solution

$$u_0 = \frac{U}{W} y + \frac{1}{2} P_x (y^2 - y), \quad (36)$$

$$w_0 = y + \frac{1}{2} P_z (y^2 - y), \quad (37)$$

The zeroth-order temperature profile

$$\varphi_0(y) = Br[-J_2 y(y-1) + J_3 y(y^2-1) + J_4 y(y^3-1)] - y. \quad (38)$$

Here J_2 , J_3 , and J_4 are constants. They encapsulate the influence of imposed boundary motion and pressure gradient within the perturbation framework. Details of the constants are given in the Appendix. Eqs. (36) and (37) represent the solutions for transverse and axial velocities of a Newtonian fluid, while Eq. (38) describes the temperature distribution.

3.2 First Order Problem $O(\epsilon^1)$ and its Solution

$$\frac{du_1}{dy} = (P_x y + A_1) \left[\left(\frac{du_0}{dy} \right)^2 + \left(\frac{dw_0}{dy} \right)^2 \right], \quad (39)$$

$$\frac{dw_1}{dy} = (P_z y + A_2) \left[\left(\frac{du_0}{dy} \right)^2 + \left(\frac{dw_0}{dy} \right)^2 \right], \quad (40)$$

$$\frac{d^2 \varphi_1}{dy^2} = - \left[\frac{d^2 \varphi_0}{dy^2} \left\{ \left(\frac{du_0}{dy} \right)^2 + \left(\frac{dw_0}{dy} \right)^2 \right\} + 2Br \left\{ \frac{du_0}{dy} \frac{du_1}{dy} + \frac{dw_0}{dy} \frac{dw_1}{dy} \right\} \right]. \quad (41)$$

The corresponding boundary conditions are:

$$\text{For } y = 0, u_1 = w_1 = 0, \text{ and } \varphi_1 = 0, \quad (42)$$

$$\text{For } y = 1, u_1 = 0, w_1 = 0, \text{ and } \varphi_1 = 0. \quad (43)$$

Furnishing a solution after integration in the form

$$u_1 = l_2(y^4 - y) + l_1(y^3 - y) + l_0(y^2 - y), \quad (44)$$

$$w_1 = M_2(y^4 - y) - M_1(y - y^3) - M_0(y - y^2). \quad (45)$$

and

$$\varphi_1(y) = y + \left[Br \left\{ j_{14} \frac{y(y^1 - 1)}{2} + \frac{j_{15}}{6} y(y^2 - 1) + j_{16}/12 y(y^3 - 1)/12 + \frac{j_{17}}{20} y(y^4 - 1) + \frac{j_{18}}{30} y(y^5 - 1) + \frac{j_{19}}{42} y(y^6 - 1) + \frac{j_{20}}{56} y(y^7 - 1) + \frac{j_{21}}{72} y(y^8 - 1) + \frac{j_{22}}{90} y(y^9 - 1) \right\} + \left\{ -\frac{j_{25}}{6} (y - y^3) - \frac{j_{26}}{12} (y - y^4) + \frac{j_{27}}{20} (y^5 - y) + \frac{j_{28}}{30} (y^6 - y) \right\} \right]. \quad (46)$$

Where $l_i, M_j, i = 0 \dots 2, j, s'$ are constant coefficients.

3.3 Second Order Problem $O(\epsilon^2)$ and its Solution

$$\frac{du_2}{dy} = (P_x y + A_1) \left[2 \frac{du_0}{dy} \frac{du_1}{dy} + 2 \frac{dw_0}{dy} \frac{dw_1}{dy} \right], \quad (47)$$

$$\frac{dw_2}{dy} = (P_z y + A_2) \left[2 \frac{du_0}{dy} \frac{du_1}{dy} + 2 \frac{dw_0}{dy} \frac{dw_1}{dy} \right], \quad (48)$$

$$\frac{d^2 \varphi_2}{dy^2} = - \left[\left(\frac{du_0}{dy} \right)^2 \frac{d^2 \varphi_1}{dy^2} + \left(\frac{dw_0}{dy} \right)^2 \frac{d^2 \varphi_1}{dy^2} + 2 \frac{d^2 \varphi_0}{dy^2} \frac{du_0}{dy} \frac{du_1}{dy} + 2 \frac{d^2 \varphi_0}{dy^2} \frac{dw_0}{dy} \frac{dw_1}{dy} + Br \left[2 \frac{du_0}{dy} \frac{du_2}{dy} + 2 \frac{dw_0}{dy} \frac{dw_2}{dy} + \left(\frac{du_1}{dy} \right)^2 + \left(\frac{dw_1}{dy} \right)^2 \right] \right] \quad (49)$$

Corresponding dimensionless boundary conditions

$$\text{At } y = 0, u_2 = w_2 = 0, \text{ and } \varphi_2 = 0, \quad (50)$$

$$\text{At } y = 1, u_2 = w_2 = 0, \text{ and } \varphi_2 = 0. \quad (51)$$

Introducing zeroth and first order solutions in the second order system (47)-(49), and after integration, we arrive at:

$$u_2 = l_3 y(y - 1) + l_4 y(y^2 - 1) + l_5 y(y^3 - 1) + l_6 y(y^4 - 1) + l_7 y(y^5 - 1), \quad (52)$$

$$w_1 = M_3 y(y - 1) + M_4 y(y^2 - 1) + M_5 y(y^3 - 1) + M_6 y(y^4 - 1) + M_7 y(y^5 - 1). \quad (53)$$

And the second-order temperature profile

$$\varphi_2(y) = Br [j_{39} y(y - 1) + j_{40} y(y^2 - 1) + j_{41} y(y^3 - 1) + j_{42} y(y^4 - 1) + j_{43} y(y^5 - 1) + j_{44} y(y^6 - 1) + j_{45} y(y^7 - 1) + j_{46} y(y^8 - 1) + j_{47} y(y^9 - 1)] - y. \quad (54)$$

Here $l_i, M_i, i = 3 \dots 7, j$'s are constant coefficients given in the appendix.

Finally, the recapitulated solutions of velocity profile x and z-direction, respectively, are acquired by merging equations (36), (44), (52), and (37), (45), (53) as:

$$u = \frac{U}{W} y + \left(\frac{1}{2} P_x + (W_i)^2 l_0 + (W_i)^4 l_3 \right) (y^2 - y) + ((W_i)^2 l_1 + (W_i)^4 l_4) (y^3 - y) + ((W_i)^4 l_2 + (W_i)^4 l_5) (y^4 - y) + (W_i)^4 l_6 (y^5 - y) + (W_i)^4 l_7 (y^6 - y), \quad (55)$$

and

$$w = y + \left(\frac{1}{2}P_z + (W_i)^2M_0 + (W_i)^4M_3\right)(y^2 - y) + \left((W_i)^2M_1 + (W_i)^4M_4\right)(y^3 - y) + \left((W_i)^4M_2 + (W_i)^4M_5\right)(y^4 - y) + (W_i)^4M_6(y^5 - y) + (W_i)^4M_7(y^6 - y), \quad (56)$$

The dimensionless temperature profile

$$\begin{aligned} \varphi(y) = 1 - Br[-J_2y(y-1) + J_3y(y^2-1) + J_4y(y^3-1)] - y - W_i^2[y + \left\{B_r \left\{j_{14}\frac{y(y^1-1)}{2} + \frac{j_{15}}{6}y(y^2-1) + j_{16}/12y(y^3-1)/12 + \frac{j_{17}}{20}y(y^4-1) + \frac{j_{18}}{30}y(y^5-1) + \frac{j_{19}}{42}y(y^6-1) + \frac{j_{20}}{56}y(y^7-1) + \frac{j_{21}}{72}y(y^8-1) + \frac{j_{22}}{90}y(y^9-1)\right\} + \left\{-\frac{j_{25}}{6}(y-y^3) - \frac{j_{26}}{12}(y-y^4) + \frac{j_{27}}{20}(y^5-y) + \frac{j_{28}}{30}(y^6-y)\right\}\right\}] + W_i^4[B_r[j_{39}y(y-1) + j_{40}y(y^2-1) + j_{41}y(y^3-1) + j_{42}y(y^4-1) + j_{43}y(y^5-1) + j_{44}y(y^6-1) + j_{45}y(y^7-1) + j_{46}y(y^8-1) + j_{47}y(y^9-1)] - y]. \end{aligned} \quad (57)$$

Here j_i are the constants. Employing equation (57) in equation (25), we acquire

$$\begin{aligned} Q(y) = 3 + \left\{\frac{j_{25}}{6}(1-3y^2) + \frac{j_{26}}{12}(1-4y^3) + \frac{j_{27}}{20}(1-5y^4) + \frac{j_{28}}{30}(1-6y^5)\right\} + Br[-c_1(1-2y) - c_2(1-3y^2) - c_3(1-4y^3) - c_4(1-5y^4) - c_5(1-6y^5) - c_6(1-7y^6) - c_7(1-8y^7) - c_8(1-9y^8) - c_9(1-10y^9)]. \end{aligned} \quad (58)$$

where c_k , ($k=1, 2, 3, \dots, 9$) are constants. It is noteworthy here that solutions achieved for velocities in equations (55) and (56) are exactly similar to the classical Newtonian fluid [17] for $W_i^2 \rightarrow 0$ and $Br \rightarrow 0$ for the temperature profile.

4. Results and Discussion

This study examines the flow characteristics of a steady, co-rotational Maxwell fluid within a helical screw. We employed the perturbation method to obtain an analytical solution for the velocity profile, temperature distribution, and heat flux. We used Mathematica 7 symbolic software to solve the mathematical model obtained from the analytical framework. This study builds upon [14], and the author provided graphical results in their work. In light of this, a comprehensive graphical analysis was performed to elucidate the effects of various dimensionless parameters on the temperature profiles

and heat flux, providing valuable insights into the fluid's behaviour.

Fig. 2(a) illustrates the non-dimensional temperature profile as a function of the depth variable y for several Weissenberg numbers (Wi^2) at specific parameter values ($P_x = -3/2$, $P_z = -3/2$, $Br = 8$, $\theta = 17.70$). The results show that increasing Wi^2 leads to a flattening of the temperature curve. This is attributed to the fluid's elasticity, which intensifies shear-thinning behaviour and reduces shear viscosity. As a result, internal heat generation due to viscous dissipation decreases, minimizing the temperature difference between the barrel wall and the screw. This phenomenon enhances the extrusion process. The Brinkman number (Br) is a critical parameter that highlights the significance of viscous dissipation relative to external heat transfer. Fig. 2(b) demonstrates the impact of Br on the dimensionless temperature distribution within the screw channel and barrel, with specific parameter values ($P_x = -3/2$, $P_z = -3/2$, $Wi^2 = 1/4$, $\theta = 17.70$). The results reveal that viscous dissipation substantially influences the temperature distribution.

An increase in Brinkman number (Br) leads to higher internal viscous heating, raising the temperature of the fluid. While this can enhance flow uniformity and reduce pressure build up, excessive viscous heating can cause overheating, leading to thermal degradation or inconsistent flow. Notably, fluid adjacent to the barrel becomes hotter compared to the region near the inner wall of the barrel (at the screw) as the Brinkman number (Br) rises from 1 to 5, consistent with heat transfer from the barrel wall to the fluid. Most polymer melts are processed safely between 180 °C and 280 °C, depending on the material. Therefore, dimensionless $\varphi(y) \leq 1.4$ corresponds to an acceptable processing temperature. If it rises (e.g., >1.5) under certain parameter conditions (high Br , Wi^2 , or pressure gradient), the dimensionless temperature may exceed 280-300 °C, risking polymer degradation and changes in colour, viscosity, or molecular weight. Additionally, the magnitude of the non-dimensional pressure gradient dictates the extent of viscous heating, highlighting the complex interplay between pressure-driven flow and thermal transport. Fig. 3(a) and Fig. 3(b) illustrate the influence of the pressure gradient on the temperature

profile, with specific parameter values ($Wi^2 = 1/4$, $Br = 4$, and $\theta = 17.70$).

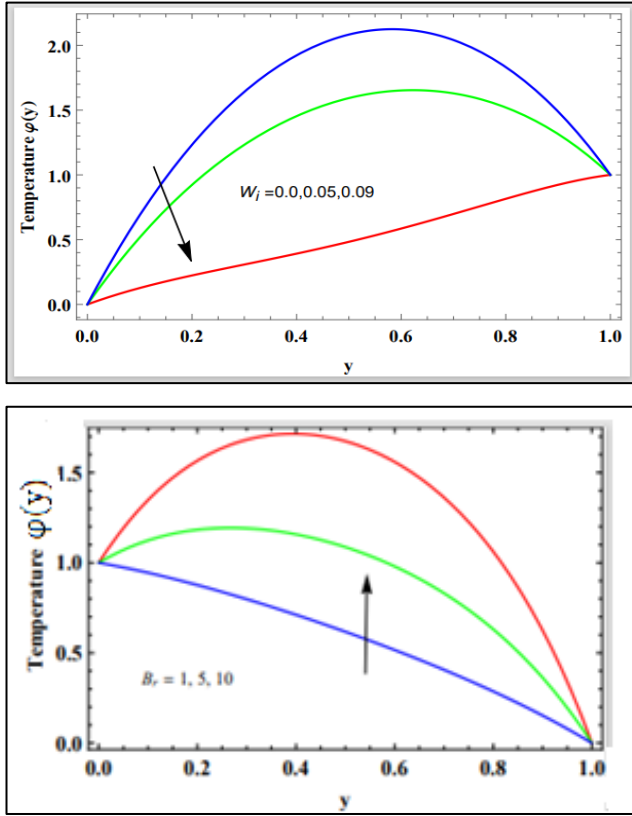


Fig. 2. Performance of (a) Wi^2 when $Br=8$ and (b) Br for $Wi^2=0.25$, fixing $P_x = -\frac{3}{2}$, $P_z = -\frac{3}{2}$, $\theta = 17.70$.

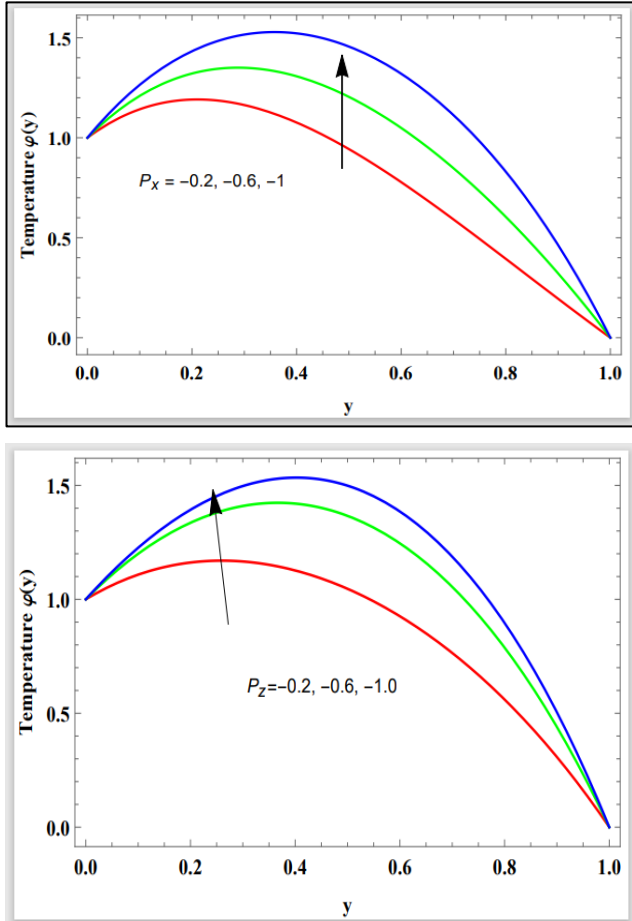


Fig. 3. Variation of (a) P_x (b) P_z on $\varphi(y)$, fixing $Wi^2 = \frac{1}{4}$, $\theta = 17.70$, $Br = 4$.

A non-zero pressure gradient P_x creates a driving force in the transverse x-direction, developing the $u(y)$ velocity profile. As P_x increases, it leads to greater shear deformation in that direction. This enhanced shear raises viscous dissipation and increases the local temperature. In contrast, the z-direction aligns with the screw and the main extrusion flow. As P_z increases, it causes stronger axial acceleration of the fluid, leading to steeper velocity gradients and significantly more viscous heating, which further raises the temperature.

The effects of the dimensionless pressure gradients P_x (transverse) and P_z (axial) on heat flux are consistent across Fig. 4(a) and 4(b). Increasing P_x leads to higher shear rates in the x-direction, which increases viscous dissipation. As a result, more internal heat is generated, leading to a rise in heat flux near the top and bottom boundaries, as mentioned in Fig. 4(a). Fig. 4(b) demonstrates that as P_z increases, the heat flux profile steepens, particularly in the lower half of the channel. The shift in the heat flux curve demonstrates that the dominant heat flow is occurring near a different wall. Figure 5 highlights the influence of Br on positive heat flux. Notably, at lower Br values, the temperature distribution exhibits a more uniform profile, indicating reduced thermal gradients. Additionally, increased viscous dissipation intensifies the temperature gradient, particularly at the screw surface.

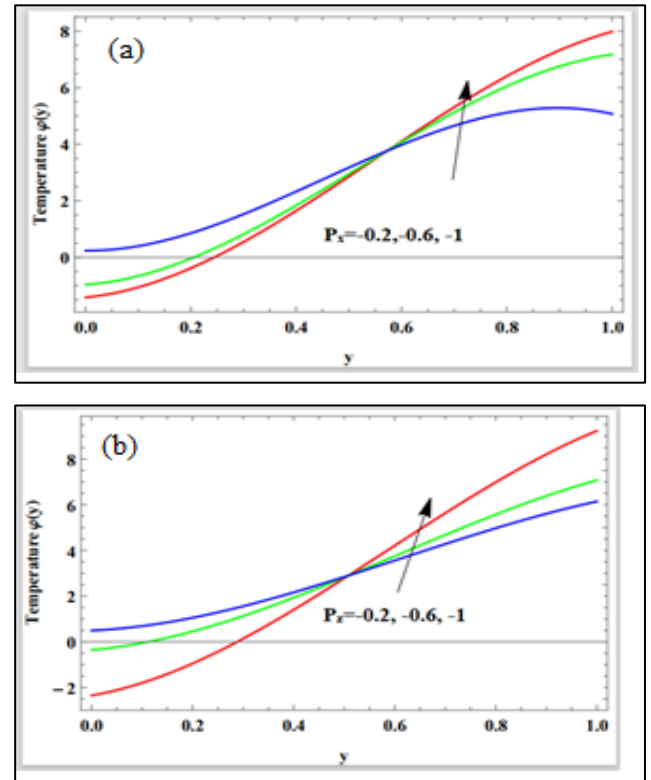


Fig. 4. Variation in (a) P_x , (b) P_z on heat flux, fixing $Br=4$, $Wi^2 = \frac{1}{4}$, $\theta = 17.70$.

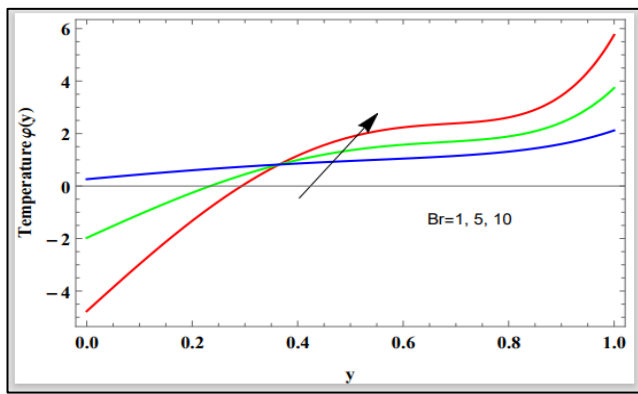


Fig. 5. Variation of Br , fixing $Wi^2 = 1/4$, $P_x = -3/2$, $P_z = -3/2$, and $\theta = 17.7^\circ$.

In helical screw flows, a larger flight angle reduces the effective axial velocity due to transverse shearing, thereby reducing the effective transport of fluid along the screw axis. This behavior stems from both the geometrical decomposition of the helical motion and the fluid-structure interaction in the extrusion process. Furthermore, the velocity in the y -direction remains unaffected due to the 2D assumption and the lubrication-type flow model used in the study.

5. Conclusion

We addressed the heat equation for the steady, incompressible flow of the co-rotational Maxwell viscoelastic model in the single-screw extruder process. The fluid model is transformed into a dimensionless form by applying appropriate non-dimensional variables. The analytical findings from the proposed model offer direct insights that can significantly aid in optimizing operating conditions in industrial extrusion processes, particularly when dealing with viscoelastic and temperature-sensitive materials like polymers. The observed trends are physically consistent, which indirectly validates the model.

The key points are:

- Brinkman number (Br) and viscous dissipation impact temperature rise, which helps designers predict hot spots within the extruder and adjust barrel heater zones to prevent thermal degradation. Thus, for lower values of the Br , $\phi(y) \leq 1.4$ is acceptable for the extrusion process.
- Smaller pressure gradients decrease shear, leading to less viscous dissipation and temperature.
- For materials with high elasticity (i.e., large Weissenberg number Wi^2), adjusting screw geometry and process parameters can help reduce problems like elastic recoil, flow instabilities, or melt fracture during extrusion.
- Heat flux increases significantly with higher pressure gradients and Brinkman number.

The model offers a quantitative tool to define safe ranges of barrel speed V , pressure gradients, and wall temperatures.

Limitations: While the model offers valuable analytical insights, it also has some limitations. For the 3D flow regime, we will not ignore curvature and inertia. The model does not account for phase transitions, which is a critical aspect in the transition zone of extrusion. In some polymer processing applications, wall slip can occur, especially at high shear rates, so one can consider it.

6. References

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Appendix:

$$l_0 = \frac{1}{2} \left(P_x + \frac{3U^2 P_x^2}{W^2} - \frac{3UP_x^2}{W} + \frac{3P_x^3}{4} + \frac{2UP_x}{W} - 2P_x P_x P_z - \frac{UP_z}{W} + \frac{3}{4} P_x P_z^2 \right), l_1 = \frac{1}{3} \left(\frac{3UP_x^2}{W} - \frac{3P_x^3}{2} + 2P_x P_z + \frac{UP_z^2}{W} - \frac{3}{2} P_x P_z^2 \right), l_2 = \frac{1}{4} (P_x^3 + P_x P_z^2),$$

$$M_0 = \frac{1}{2} \left(\frac{2UP_x}{W} - P_x^2 + 3P_z + \frac{U^2 P_z}{W^2} - \frac{2UP_x P_z}{W} + \frac{3P^2 P}{4} - 3P_x^2 P_z - 3P_z^2 + \frac{3}{4} P_z^3 \right), M_1 = \left(P_x^2 + \frac{2UP_x P_z}{W} - 3P_x^2 P_z + 3P_z^2 - \frac{3}{2} P_z^3 \right), M_2 = (P_x^2 P_z + P_z^3),$$

$$J_2 = \frac{1}{2} \left\{ \left(\frac{U}{W} \right)^2 - \frac{U}{W} P_x + \frac{1}{4} P_x^2 + 1 - P_z - \frac{1}{4} P_z^2 \right\},$$

$$J_3 = \frac{\left\{ \frac{2U}{W} P_x - P_x + 2P_z - P_z^2 \right\}}{2.3}, J_4 = \frac{\{P_x^2 + P_z^2\}}{3.4},$$

$$j_7 = l_0^2 + l_1^2 + l_2^2 + 2l_0 l_1 + 2l_0 l_2 + 2l_1 l_2 + M_0^2 + M_1^2 + M_2^2 + 2M_0 M_1 + 2M_0 M_2 + 2M_1 M_2, j_8 = -4l_0^2 - 4l_0 l_1 - 4l_0 l_2 - 4M_0^2 - 4M_0 M_1 - 4M_0 M_2, j_9 = 4l_0^2 - 6l_1^2 - 6l_0 l_1 - 6l_1 l_2 + 4M_0^2 - 6M_1^2 - 6M_0 M_1 - 6M_1 M_2,$$

$$j_{10} = -8l_2^2 + 12l_0 l_1 - 8l_0 l_2 - 8l_1 l_2 - 8M_2^2 + 12M_0 M_1 - 8M_0 M_2 - 8M_1 M_2,$$

$$j_{11} = 9l_1^2 + 16l_0 l_2 + 9M_1^2 + 16M_0 M_2,$$

$$j_{12} = (24l_1 l_2 + 24M_1 M_2), j_{13} = 16l_2^2 + 16M_2^2,$$

$$j_{14} = 2j_2 j_7, j_{15} = 2j_2 j_8 + 6j_3 j_7, j_{16} = 2j_2 j_9 + 6j_3 j_8 + 12j_4 j_7, j_{17} = 2j_2 j_{10} + 6j_3 j_9 + 12j_4 j_8,$$

$$j_{18} = 2j_2 j_{11} + 6j_3 j_{10} + 12j_4 j_9, j_{19} = 2j_2 j_{12} + 6j_3 j_{11} + 12j_4 j_{10}, j_{20} = 2j_2 j_{13} + 6j_3 j_{12} + 12j_4 j_{11},$$

$$j_{21} = 6j_3 j_{13} + 12j_4 j_{12}, j_{22} = 12j_4 j_{13}, j_{25} = \left[4 \frac{U}{W} l_0 - 2P_x l_0 + 4M_0 - 2P_z M_0 \right]$$

$$j_{26} = \left[4P_x l_0 + 3 \frac{U}{W} l_1 - 3P_x l_1 + 6M_1 + 4P_z M_0 - 3P_z M_1 \right],$$

$$j_{27} = \left[6P_x l_0 + 3 \frac{U}{W} l_1 - 3P_x l_1 + 8M_2 + 6P_z M_1 - 4P_z M_2 \right], j_{28} = [8P_x l_2 + 4P_z M_2], j_{29} =$$

$$\frac{2}{3} j_2 j_{25} j_{30} = \frac{1}{3} j_2 j_{26} + 2j_3 j_{25} j_{31} = \frac{1}{5} j_2 j_{27} + j_3 j_{26} +$$

$$4j_4 j_{25} j_{32} = \frac{2}{15} j_2 j_{28} + \frac{3}{5} j_3 j_{27} + 4j_4 j_{26}, j_{33} = \frac{2}{5} j_3 j_{28} + \frac{6}{5} j_4 j_{27} j_{34} = \frac{4}{4} j_4 j_{28}, j_{35} = 4[l_0^2 + M_0^2], j_{36} =$$

$$12[l_0 l_1 + M_0 M_1], j_{37} = 16[l_0 l_2 + M_0 M_2] + 9[l_1^2 + M_1^2], j_{38} = 24[l_1 l_2 + M_1 M_2], j_{39} = 16[l_2^2 + M_2^2], j_{40} =$$

$$\begin{aligned}
\frac{1}{2}[j_{14}], j_{41} &= \frac{1}{6}[j_{15}], j_{42} = \frac{1}{12}[j_{16}-2j_{35}], j_{43} = \frac{1}{20}[j_{17} + \\
j_{29}-2j_{36}-\frac{1}{3}j_{25}], j_{44} &= \frac{1}{30}[j_{18} + j_{30}-2j_{37}-\frac{1}{6}j_{26}] \\
j_{45} &= \frac{1}{42}[j_{19} + j_{31}-2j_{38}-\frac{1}{10}j_{27}], \quad j_{46} = \frac{1}{56}[j_{20} + \\
j_{32}-2j_{39}-\frac{1}{15}j_{28}], \quad j_{47} &= \frac{1}{72}[j_{21} + j_{33}], \quad j_{48} = \frac{1}{90}[j_{22} + \\
j_{34}], \quad c_1 &= [j_2 + \frac{1}{2}j_{14}-j_{40}], \quad c_2 = [j_3 + \frac{1}{6}j_{15}-j_{41}], \quad c_3 = \\
[j_4 + \frac{1}{12}j_{16}-j_{42}], \quad c_4 &= [\frac{1}{20}j_{17}-j_{43}], \\
c_5 &= [\frac{1}{30}j_{18}-j_{44}], \quad c_6 = [\frac{1}{42}j_{19}-j_{45}], \quad c_7 = \\
[\frac{1}{56}j_{20}-j_{46}], \quad c_8 &= [\frac{1}{72}j_{21}-j_{47}], \quad c_9 = [\frac{1}{90}j_{22}-j_{48}]
\end{aligned}$$

NOMENCLATURE

ρ constant fluid density (kg/m³)

$\mathbf{V}$ barrel velocity (m/s)

μ dynamic fluid viscosity (Pa.s=kg/m.s)

$\mathbf{f}$ body force (N)

$\mathbf{T}$ stress tensor (Pa)

$\mathbf{T}$ Temperature (Unit: K or °C)

P dynamic pressure (N/m)

Q heat flux (W/m²)

k thermal conductivity (W/m.K)

B channel width (m)